MATHEMATICS

B.Sc. Part II, Paper III, Topic - Comparison Test (2nd type)

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(i) If $\sum u_n$ and $\sum v_n$ be two positive term series and after some particular term $\frac{u_n}{u_{n-1}} < \frac{v_n}{v_{n-1}}$, then $\sum u_n$ converges provided that $\sum v_n$ converges.

Proof: Let the series begining after some particular term be

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \text{ to } \infty,$$

and $v_1 + v_2 + v_3 + \cdots + v_n + \cdots$ to ∞.

It is given that

$$\frac{u_2}{u_1} < \frac{v_2}{v_1}$$

$$\frac{u_3}{u_2} < \frac{v_3}{v_2}$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\frac{u_n}{u_{n-1}} < \frac{v_n}{v_{n-1}}.$$

Multiplying all these inequalities, we get

$$\frac{u_2}{u_1} \cdot \frac{u_3}{u_2} \cdots \frac{u_n}{u_{n-1}} < \frac{u_2}{u_1} \cdot \frac{v_3}{v_2} \cdots \frac{v_n}{v_{n-1}}$$

or $\frac{u_n}{u_1} < \frac{v_n}{v_4}$; or $u_n < \left(\frac{u_1}{v_1}\right) v_n$.

Let $\frac{u_1}{v_1} = k$. Then $u_n < k v_n$; or $\sum u_n < k \sum v_n$.

If $\sum v_n$ converges, then $\sum u_n$ also converges.

(ii) If $\sum u_n$ and $\sum v_n$ be two positive term series and after some particular term $\frac{u_n}{u_{n-1}} > \frac{v_n}{v_{n-1}}$, then $\sum u_n$ diverges provided that $\sum v_n$ diverges.

Proof : Let the series beginning after some particular term be

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \text{ to } \infty$$

and $v_1 + v_2 + v_3 + \cdots + v_n + \cdots$ to ∞.

It is given that

$$\frac{u_2}{u_1} > \frac{v_2}{v_1}$$

$$\frac{u_3}{u_2} > \frac{v_3}{v_2}$$

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$$\text{-------}$$

$$\frac{u_n}{u_{n-1}} > \frac{v_n}{v_{n-1}}$$

Multiplying all these inequalities, we get

$$\frac{u_2}{u_1} \cdot \frac{u_3}{u_2} \cdots \frac{u_n}{u_{n-1}} > \frac{v_2}{v_1} \cdot \frac{v_3}{v_2} \cdots \frac{v_n}{v_{n-1}}$$

or $\frac{u_n}{u_1} > \frac{v_n}{v_1}$; or $u_n > \left(\frac{u_1}{v_1}\right) v_n$.

Let $\frac{u_1}{v_1} = K$.

Then $u_n > K v_n$; or $\sum u_n > K \sum v_n$.

If $\sum v_n$ is divergent, then $\sum u_n$ is also divergent.

Note This test is useful in ~~providing~~ proving higher tests such as Raabe's test, etc. If De'almbert ratio test fails then we apply Raabe's test.

Theorem , Raabe's test

Statement : The positive term series $\sum u_n$ converges or diverges according as $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right) > \text{or} < 1.\right.$

Proof. Let us compare the given series $\sum u_n$ with the auxilliary series $\sum v_n$,

where $\sum v_n = \sum \frac{1}{n^p} + \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots$ to ∞

We know that $\sum v_n$ is convergent or divergent according as $p >$ or < 1.

Hence, by comparison test, the given series $\sum u_n$ will be convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{or} < \frac{v_n}{v_{n+1}}$$

$$\text{or} \quad \frac{u_n}{u_{n+1}} > \text{or} < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\text{or} \quad \frac{u_n}{u_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)^p$$

$$\text{or} \quad \frac{u_n}{u_{n+1}} > \text{or} < 1 + p\cdot\frac{1}{n} + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n^2} + \cdots$$

by Binomial Theory

$$\text{or} \quad \frac{u_n}{u_{n+1}} - 1 > \text{or} < \frac{1}{n}\left[p + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n} + \cdots\right]$$

$$\text{or} \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) > \text{or} < p + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n} + \cdots$$

Proceeding to limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > \text{or} < p$$

or $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} >$ or < 1.

$\therefore$ Hence, the theorem.

<u>Remark</u>

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = 1,$$ then Raabe's test fails.

Then we proceed for further tests.

<u>Example</u> Find whether the series

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1.3}{2.4}\,\frac{x^3}{5}+\frac{1.3.5\,x^4}{2.4.6.7}+\cdots \text{ to } \infty, (x>0)$$

is convergent or divergent.

<u>Solution</u>: Here $u_n = \frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}\cdot\frac{x^n}{2n-1}$

and $u_{n+1} = \frac{1.3.5\ldots.(2n-3)(2n-1)}{2.4.6\ldots.(2n-2)(2n)}\cdot\frac{x^{n+1}}{2n+1}$.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n}{2n-1}\cdot\frac{2n+1}{2n-1}\cdot\frac{x^n}{x^{n+1}} = \frac{2}{2-\frac{1}{n}}\cdot\frac{2+\frac{1}{n}}{2-\frac{1}{n}}\cdot\frac{1}{x}.$$

$$\therefore \mathop{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{2}{2}\cdot\frac{2}{2}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{1}{x}>1$ i.e. $x<1$ and diverges when $\frac{1}{x}<1$ i.e. $x>1$.

when $x=1$, $\mathop{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$ and the ratio test fails.

when $x=1$, $\frac{u_n}{u_{n+1}} = \frac{(2n)(2n+1)}{(2n-1)(2n-1)} = \frac{4n^2+2n}{4n^2-4n+1}$

i.e. $n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{4n^2+2n}{4n^2-4n+1}-1\right) = \frac{n(6n-1)}{4n^2-4n+1}$

$$= \frac{6-\frac{1}{n}}{4-\frac{4}{n}+\frac{1}{n^2}}$$

$\therefore \mathop{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{6}{4} = \frac{3}{2}$ which is greater than unity.

$\therefore$ By Raabe's test, $\sum u_n$ is convergent when $x=1$.

Hence $\sum u_n$ is convergent when $x\le 1$ and divergent when $x>1$.